Least Upper Bounds

Recall the definitions of upper bound and least upper bound.

Definition. A set $A$ of real numbers is bounded above if there is a number $x$ such that
\[ x \geq a \quad \text{for every } a \text{ in } A. \]
Such a number $x$ is called an upper bound for $A$.

Definition. A number $x$ is a least upper bound for a set $A$ if
\[
\begin{aligned}
& x \text{ is an upper bound for } A, \\
& \text{if } y \text{ is an upper bound for } A, \text{ then } x \leq y.
\end{aligned}
\]
Such a number $x$ is also called the supremum for $A$ and sometimes denoted by $\sup A$ or $\text{lub } A$.

There is an equivalent definition of least upper bound.

Definition. A number $x$ is a least upper bound for a set $A$ if
\[
\begin{aligned}
& x \text{ is an upper bound for } A, \\
& \text{for every } \epsilon > 0, \text{ there is an } x_{\epsilon} \in A \text{ such that } x - \epsilon < x_{\epsilon} \leq x.
\end{aligned}
\]
Such a number $x$ is also called the supremum for $A$ and sometimes denoted by $\sup A$ or $\text{lub } A$.

To show that the two definitions are equivalent, we must prove the following If and Only If Theorem:

Theorem. If $x$ is an upper bound for $A$, then
\[ x \leq y. \]
if and only if
\[ \text{for every } \epsilon > 0, \text{ there is an } x_{\epsilon} \in A \text{ such that } x - \epsilon < x_{\epsilon} \leq x. \]

Proof: First $(2) \Rightarrow (2')$. Assume $(2)$. The proof is by contradiction. Assume there IS an $\epsilon > 0$ such that there is no $x_{\epsilon} \in A$ such that $x - \epsilon < x_{\epsilon} \leq x$. But then $x - \epsilon$ would be an upper bound for $A$ which is less than $x$.

Second $(2') \Rightarrow (2)$. Again use contradiction. If $(2)$ is false and $(2)$ is true, there is an upper bound $b$ for $A$ which satisfies $b < x$. Let $\epsilon = x - b > 0$. There is no $x_{\epsilon} \in A$ such that $x - \epsilon = b < x_{\epsilon} \leq x$. 