MthT 430 Notes Chapter 8d Least Upper Bounds

Least Upper Bounds

Recall the definitions of *upper bound* and *least upper bound*.

Definition. *A set A of real numbers is* **bounded above** *if there is a number x such that*

$$x \geq a \quad \text{for every } a \text{ in } A.$$

Such a number x is called an **upper bound** *for A.*

Definition. *A number x is a* **least upper bound** *for a set A if*

$$x \text{ is an upper bound for } A, \tag{1}$$

$$\text{if } y \text{ is an upper bound for } A, \text{ then } x \leq y. \tag{2}$$

Such a number x is also called the **supremum** *for A and sometimes denoted by* $\sup A$ *or* lub$A$.

There is an equivalent definition of *least upper bound*.

Definition. *A number x is a* **least upper bound** *for a set A if*

$$\begin{cases} x \text{ is an upper bound for } A, & (1) \\ \text{For every } \epsilon > 0, \text{ there is an } x_\epsilon \in A \text{ such that } x - \epsilon < x_\epsilon \leq x. & (2') \end{cases}$$

Such a number x is also called the **supremum** *for A and sometimes denoted by* $\sup A$ *or* lub A.

To show that the two definitions are equivalent, we must prove the following *If and Only If Theorem*:

Theorem. *If* x is an upper bound for A, *then*

$$\text{If } y \text{ is an upper bound for } A, \text{ then } x \leq y. \tag{2}$$

if and only if

$$\text{For every } \epsilon > 0, \text{ there is an } x_\epsilon \in A \text{ such that } x - \epsilon < x_\epsilon \leq x. \tag{2'}$$

Proof: First (2) $\Rightarrow$ (2′). Assume (2). The proof is by contradiction. Assume there IS an $\epsilon > 0$ such that there is no $x_\epsilon \in A$ such that $x - \epsilon < x_\epsilon \leq x$. But then $x - \epsilon$ would be an upper bound for A which is less than x.

Second (2′) $\Rightarrow$ (2). Again use contradiction. If (2) is false and (2) is true, there is an upper bound b for A which satisfies $b < x$. Let $\epsilon = x - b > 0$. There is no $x_\epsilon \in A$ such that $x - \epsilon = b < x_\epsilon \leq x$.