Exponential Functions

<u>Example</u> GDP grows at 4% per year.

If population is now 1.4 (trillion) what is GDP after 1 yr., 23 years, 23 and $\frac{1}{2}$ years

$G(0) = 1.4$

$G(1) = 1.4\,(1+.04)$

$G(2) = 1.4\,(1+.04)^2$

$G(23) = 1.4\,\underbrace{(1+.04)^{23}}_{\text{multiply by } 1.04 \text{ 23 times}}$

Exponential Function (equal changes in (argument) lead to equal ~~changes~~ multiplication)

If $b>0$

$$f(x+\Delta x) = f(x)\,C(\Delta x)$$

for all x

Ofcourse functions: For $b>0$

b^x : b^n, $n=1,2,\ldots = \underbrace{b \cdot \ldots \cdot b}_{\text{multiply } n \text{ times}}$

$n=0$

convention $b^0 = 1$ $\quad b^{1/2} = \sqrt{b}$

Fractional (Roots)

($b>0$) $\quad f(x) = b^x$

b — positive integer

("power functions" of x $\quad f(x) = x^{b \leftarrow \text{fixed}}$

Exponential $\quad f(x) = b^x$ (x changes (check) x = two all x, pos, neg, 0]

$\uparrow$ fixed

[$\boxed{\text{3 RULES}}$] p.291 $\quad b^x = b^y$ iff $x=y$

$\boxed{b>0}$

$(b^x)(b^y) = b^{(x+y)}$

($\quad \cancel{b^x \div b}\; b^{-x} = \dfrac{1}{b^x}$ [$\dfrac{b^x}{b^y} = b^{x-y}$

First solving equation. (Ex. 4.1, 31)

$$5^{x^2+2x} = 125 = 5^3 \text{ (just remember!)}$$

$x^2+2x = 3$ (two roots cu

$x^2 + \cancel{2x} - 3 = 0$ $\cancel{x=}$ $(x+3)(x-1)=0$

$$\boxed{x=0,\ x=-3}$$

SPECIAL LIMIT

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \quad \text{(RECALL)}$$

COMPOUND INTEREST

Start with p Annual r (decimal)

After t years think integer $p(1+r)^t$

<u>Half year</u> $p(1+\frac{1}{2}r)$

<u>One year</u> $p(1+\frac{r}{2})^2$ compounding; i.e. pay "interest on interest"

What is factor; comp semi annually

$$B(t) = p\left(1+\frac{r}{2}\right)^{2t}$$

<u>The trick</u> Compound "k times annually"

one $k=2$, $k=4$, "daily"

Each year multy by $(1+r)$ $(1+\frac{r}{2})^2$ $(1+\frac{r}{4})^4$, $(1+\frac{r}{360})^{360}$

The Algebraic Trick

$$\left(1+\frac{r}{k}\right)^{kt} = \left(1+\frac{r}{k}\right)^{\left[\frac{k}{r}\right]\cdot rt}$$

$\frac{k}{r} = N$ large if k is large

$$\left[\left(1+\frac{1}{N}\right)^N\right]^{rt}, \quad \lim_{N\to\infty} e^{(r)t}$$

cont
We say that interest is "Compounded Continuously" (CC) meaning –
process : compound a bunch of times let # times → ∞ one short the period

<u>Compare</u> 5% sy $(Y_1)(1) = 1.05$

1.05 $\quad Y_1(2) = 1.~~5125~~$ 1.0506

$(1+\frac{.05}{2})$ $\quad = ~~1.05116~~$

4 2 $\quad$ 12 $\quad$ 1.05116...

12 $\quad$ ↑ gain of 1%

100 $\quad$ 1.05125...

360 $\quad$ 1.05126 7446

$e^{(.05)}$ 1.0512710896

<u>close</u> !

Exponential Growth and Decay

Growth $\quad Q_0 e^{kt}$

$$Q_0 e^{k(t+1)} = Q_0 [e^{kt} \cdot e^{k}] \; ; \; Q_0 b^t, \; b > 1$$

$$= \text{multiply by } e^k \qquad Q_0 [e^k]^t$$

↑ b

4.1.9 Culture grows exponentially

one hour $1000

10 years annual rate 7%

Present Value

Optimization 20090305 sp

(2009 testtwosample #15

Maximize xy^2, $\underbrace{x+y=10,\ x>0,\ y>0}_{\text{CONSTRAINT}}$

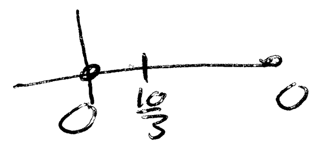
$y=10-x$

$$F(x,y) = xy^2 = x(10-x)^2$$

$$\frac{dF}{dx} = 1\cdot(10-x)^2 - 2x(10-x)$$

$$= (10-x)(10-x-2x)$$

$x=\frac{10}{3}$ (also $x=10$



$$y=10-\frac{10}{3}=\frac{20}{3}$$

Alternate $x=10-y$

$$F(y) = x\,y^2 = (10-y)y^2 = 10y^2-y^3$$

$$\frac{dF}{dy} = 20y-3y^2 = y(20-3y)$$

Again $y=0$ is "END"