8.2, #23. The equation of the ellipse is $(x/2)^2 + (y)^2 = 1$. The upper half of the ellipse is the graph of $f(x) = \sqrt{1-(x/2)^2}$. We compute $1 + f'(x)^2 = 1 + \dfrac{x^2}{1-(x/2)^2}$. The circumference is given by

$$4\int_0^2 \sqrt{\frac{1-3x^2/16}{1-x^2/4}}\,dx.$$

The integral is improper; the integrand has an infinite limit at $x = 2$. This is because the ellipse has a vertical tangent at $(2, 0)$. One solution is to integrate from 0 to c for $c < 2$ and take different values of c getting closer to 2. A better method is to make a change of variables which changes the integral to a proper integral. Let $x = 2\sin\theta$. The integral becomes

$$4\int_0^{\pi/2} \sqrt{4\cos^2\theta + \sin^2\theta}\,d\theta.$$

The circumference is approximately 9.688.

8.3, #1. Slice the water in the tank by planes perpendicular to the axis of the cylinder. A slice has the shape of a disk of radius 6 feet and thickness Δy. In particular, the slice has weight $(62.4)(36\pi)\,\Delta y$. If the slice is y feet above the bottom of the tank, then the work pumping it $20 - y$ feet to the top is $(62.4)(36\pi)(20-y)\,\Delta y$. The work pumping all of the water to the top is

$$\begin{aligned}(62.4)(36\pi)\int_0^{10}(20-y)\,dy &= 7057.27(20y - (y^2/2))\Big|_0^{10} \\ &= 7057.27(200-50) = 1,058,591.06 \text{ foot-pounds.}\end{aligned}$$

8.3, #2. Consider slices as in #1. The work needed to move a slice y feet above the bottom of the tank to a point 10 ft above the top of the tank is $(62.4)(36\pi)(30-y)\,\Delta y$. The total work pumping all of the water to that point is

$$\begin{aligned}(62.4)(36\pi)\int_0^{20}(30-y)\,dy &= 7057.27(30y-(y^2/2))\Big|_0^{20}\\ &= 7057.27(600-200)\\ &= 2,822,909.49 \text{ foot-pounds.}\end{aligned}$$

8.3, #4. Assume the crane's motor is located on a platform 75 feet above the ground. The crane lifts the 500 lb of cement and the wieght of the cable, but as the cement rises, there is less cable to lift. When the cement is at height h there is $(75-h)$ feet of cable at 5 lb/ft and 500 lb of cement for a total of $500+(75-h)5$ lb to be lifted. The work to lift this a distance Δh ft is $\Delta W = (500+(75-h))\Delta h$ ft-lb. The total work is

$$W = \int_0^{30} 500+(75-h)5\,dh = 24000 \text{ ft-lb.}$$

8.3, #6. Slice the tank by planes parallel to the ground. So slices have rectangular cross-sections of length 12 feet and varying width. For example, the width of the topmost slice 10 feet below the ground is 0 feet, the width of the slice 14 feet below the ground is 8 feet. We introduce coordinates so that slices are $14 + y$ feet below the ground, where $-4 \le y \le 4$. Here y is the distance from the axis of the cylinder with distance increasing in the downward direction. A slice $14+y$ feet below the ground then has width $2\sqrt{16-y^2}$ feet and volume $(12)(2\sqrt{16-y^2})\,\Delta y$. The work pumping it $14+y$ feet to ground level is $(42)(12)(2\sqrt{16-y^2})(14+y)\,\Delta y$. The total work emptying the tank is

$$\int_{-4}^{4} (42)(12)(2)\sqrt{16-y^2}(14+y)\,dy.$$

Two definite integrals (disregarding multiplicative constants) need to be evaluated:

$$\int_{-4}^{4} \sqrt{16-y^2}\,dy, \qquad \int_{-4}^{4} \sqrt{16-y^2}y\,dy.$$

The first is the area of the top half of a circle of radius r and thus has value 8π. The second has an odd function as integrand (a function $f(y)$ such that $f(-y) = -f(y)$) and thus has value 0. The total work turns out to be

$$(42)(12)(2)(14)(8\pi) = 354,673.24 \text{ foot-pounds.}$$

8.3, #8. The section on the earth's gravitational field on pages 440–441 is useful for doing this problem. At a distance of r meters from the surface of the earth, the gravitational force of earth on the satellite is

$$\frac{GMm}{r^2} = (6.67\times 10^{-11}) \times (6\times 10^{24}) \times (1000)r^{-2} = (4.002\times 10^{17})r^{-2} \text{ newtons.}$$

The work needed in moving the satellite an additional distance of Δr meters is

$$\frac{GMm}{r^2} \times \Delta r = (4.002\times 10^{17})r^{-2}\Delta r \text{ newton-meters.}$$

Thus the total work needed to move the satellite a distance of 2×10^6 meters from the surface of the earth is

$$\begin{aligned}\int_{6.4\times 10^6}^{8.4\times 10^6} (4.002\times 10^{17})r^{-2}\,dr &= 4.002\times 10^{17}\left(\frac{-1}{r}\right)\Bigg|_{6.4\times 10^6}^{8.4\times 10^6}\\ &= 1.489\times 10^{10} \text{ newton-meters.}\end{aligned}$$