7.9, #21. If $x \geq 3$, then $x^2 \geq 3x$ and $e^{-x^2} \leq e^{-3x}$. Thus

$$\int_3^\infty e^{-x^2}\,dx \leq \int_3^\infty e^{-3x}\,dx = -\frac{1}{3}\lim_{b\to\infty} e^{-3x}\Big|_3^b = \frac{e^{-9}}{3}.$$

Similarly if $x \geq n$, then $x^2 \geq nx$ and $e^{-x^2} \leq e^{-nx}$. Thus

$$\int_n^\infty e^{-x^2}\,dx \leq \int_n^\infty e^{-nx}\,dx = -\frac{1}{n}\lim_{b\to\infty} e^{-nx}\Big|_n^b = \frac{e^{-n^2}}{n}.$$

7.9, #22. $\dfrac{2x^2+1}{4x^4+4x^2-2}$ behaves like $\dfrac{2x^2}{4x^4} = \dfrac{1}{2x^2}$ for large x. By the p-test integral (a) should converge while integral (b) should diverge (since $(1/2x^2)^{1/4} = 1/2^{1/4}x^{1/2}$). Indeed, $4x^2 - 2 \geq 0$ for $x \geq 1$, so

$$\begin{aligned}\int_1^\infty \frac{2x^2+1}{4x^4+4x^2-2}\,dx &\leq \int_1^\infty \frac{2x^2+1}{4x^4}\,dx \\ &= \int_1^\infty \frac{dx}{2x^2} + \int_1^\infty \frac{dx}{4x^4} \\ &= \frac{1}{2} + \frac{1}{12} = \frac{7}{12},\end{aligned}$$

using the antiderivatves $-1/2x$ and $-1/12x^3$ to calculate the two integrals. On the other hand,

$$\int_1^\infty \left(\frac{2x^2+1}{4x^4+4x^2-2}\right)^{1/4} dx \geq \int_1^\infty \left(\frac{2x^2}{4x^4+4x^4}\right)^{1/4} dx$$

since $2x^2+1 > 2x^2$ and $4x^4+4x^2-2 < 4x^4+4x^4$ for $x \geq 1$. The last integral diverges by the p-test.

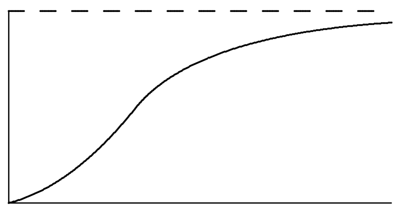
7.9, #25. The tangent line approximation $1+t$ to e^t at $t=0$ lies below the graph of e^t since the graph of e^t is concave-up. Thus $1+t \leq e^t$ for all t. Substituting $t = 1/x$ in this inequality gives $1+(1/x) \leq e^{1/x}$ or $e^{1/x} - 1 \geq (1/x)$. Thus

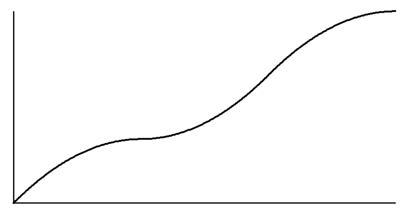
$$\int_1^\infty \frac{dx}{x^5(e^{(1/x)}-1)} < \int_1^\infty \frac{x}{x^5}\,dx$$

The second integral converges by the p-test, so the first one converges as well.

7.10, #2: $F(x)$ starts at $(0,0)$ and is increasing, concave-up on its initial segment, concave-down on its second segment, where it approaches a horizontal asymptote.

7.10, #3. $F(x)$ starts at $(0,0)$ is increasing, concave-down on its initial segment, concave-up on its middle segment, concave-down on its last segment.





7.10, #6. Note the following features of the six slope fields: (I) and (III) have horizontal slopes for large $|x|$. (II) and (IV) have steep slopes for large $|x|$ with (IV) having almost vertical slopes. Lastly, (V) and (VI) are the only slope fields with negative slopes. It follows easily that (d) $e^{-0.5x}\cos x$ is matched with (V) and (f) $-e^{-x^2}$ with (VI). The difference between (I) and (III) is that of scale with respect to the x-direction. So (b) e^{-2x^2} and (c) $e^{-x^2/2}$ are matched respectively with (I) and (III). Finally, (a) e^{x^2} is matched with (IV) and (e) $(1/(1+0.5\cos x)^2$ with (II). Indeed, since $1/2 \le 1+0.5\cos x \le 3/2$, it follows that $4/9 \le 1/(1+0.5\cos x)^2 \le 4$. Thus the steepness of the slope field of (e) is bounded, and the match must be (e) with (II) and (a) with (IV).

7.10, #8. $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{3+\cos(t^2)}\,dt = \sqrt{3+\cos(x^2)}$.

7.10, #10. $\dfrac{d}{dx}\displaystyle\int_{0.5}^x \arctan(t^2)\,dt = \arctan(x^2)$.