Limits and Order

For functions of a real variable, the derivative is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

which means that the difference

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} - f'(x)$$

is small if $\Delta x$ is small and not 0 (for which the quotient is not obviously defined).

Multiplying the remainder by $\Delta x$, we obtain that

$$f(x + \Delta x) - f(x) - f'(x)\Delta x = \text{small} \cdot \Delta x,$$

with the right hand side, (RHS), of the equation is “much smaller than $\Delta x$”, as $\Delta x \to 0$, in the precise sense

$$\lim_{\Delta x \to 0} \frac{\text{RHS}}{|\Delta x|} = 0.$$

Another formal advantage is that the equation is also defined and true for $\Delta x = 0$.

**Definition.** As $\Delta x \to 0$, an expression $\phi(\Delta x)$ is little $o$ of $\Delta x$, written $o(\Delta x)$, if

$$\lim_{\Delta x \to 0} \frac{\phi(\Delta x)}{|\Delta x|} = 0.$$

If we are not worried about the particular details of $\phi(x)$, we write $\phi(x) = o(\Delta x)$. With this convention, the definition of differentiability and the derivative takes the convenient form

$$f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x + o(\Delta x).$$

In a similar way, if $\lim_{\Delta x \to 0} \psi(\Delta x) = 0$, we write $\psi(\Delta x) = o(1)$ with the precise meaning that

$$\lim_{\Delta x \to 0} \frac{\psi(\Delta x)}{1} = 0.$$

**Definition.** Let $q(\Delta x)$ be nonzero for $\Delta x$ near 0. Then a function $\phi(\Delta x)$ is little $o$ of $q(\Delta x)$, written $\phi(\Delta x) = o(q(\Delta x))$, if

$$\lim_{\Delta x \to 0} \frac{\phi(\Delta x)}{|q(\Delta x)|} = 0.$$
Then a function \( \phi(x) \) is big \( O \) of \( q(\Delta x) \), written \( \phi(\Delta x) = O(q(\Delta x)) \), if

\[
\frac{\phi(\Delta x)}{|q(\Delta x)|}
\]

is bounded as \( \Delta x \to 0 \).

With this convention, continuity of a function \( f(x) \) can be expressed by

\[
f(x + \Delta x) = f(x) + o(1),
\]

and local boundedness of a function can be expressed as \( f(x + \Delta x) = O(1) \).

There is a formal calculus for handling sums and products for functions which are little \( o \) or big \( O \) of one (or several) \( q \). Verify that \( O(1) \cdot o(\Delta x) = o(\Delta x) \); i.e., the product of a bounded function and a function which is \( o(\Delta x) \) is \( o(\Delta x) \). Similarly \( o(\Delta x) \pm o(\Delta x) = o(\Delta x) \).

The concepts little \( o \) and big \( O \) are also useful as the argument \( x \to \infty \). For example we write \( x^2 = o(e^x) \) as \( x \to \infty \) with the precise meaning

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = 0.
\]